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Critical behaviour of an isotropic spin system. II

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Abstract. The critical behaviour of the step model of phase transitions on the square, simple cubic and body-centred cubic lattices is investigated by the method of exact series expansions. The model is found to be in agreement with two aspects of universality, namely the dependence of critical exponents on spin-space dimensionality and the lattice independence of critical exponents for fixed lattice and spin-space dimensionality. Series expansions for the three-dimensional susceptibility and specific heats are generated and analysed, and the critical exponents are found to be $\gamma = 1{\cdot}335 \pm 0{\cdot}01$ and $\alpha = -0{\cdot}04 \pm 0{\cdot}10$ respectively. In two dimensions the existence and nature of a phase transition remains an open question. No evidence of either a conventional algebraic singularity or a vortex induced essential singularity is found.

1. Introduction

In Guttmann *et al* (1972) a new model exhibiting a phase transition was introduced. This model, called the 'step model', combined certain features of both the $S=\frac{1}{2}$ Ising and the planar classical Heisenberg (PCH) model. Series expansions for the model were generated and analysed for the face-centred cubic (FCC) and triangular (T) lattices in Guttmann and Joyce (1973, to be referred to as I).

The principal purpose of introducing the model was to test one aspect of the universality hypothesis which is that the critical exponents should be characterised by the symmetry of the order parameter, other characteristic parameters being equal. In this paper we have generated and analysed series for the step model on a simple cubic (SC), body-centred cubic (BCC) and a square (SQ) lattice. This enables us to further test the above aspect of the universality hypothesis, and also to test another aspect, that of lattice independence for a given dimensionality.

With the recent interest in two-dimensional planar spin systems, square lattice series might also be expected to shed some light on the nature of the phase transition for this system. In fact, it serves to demonstrate the richness of critical behaviour for models of this class, in that it seems to behave neither like the Ising model nor like the PCH model, that is, the susceptibility shows no sign of a conventional algebraic singularity, as observed for the Ising model, nor does it show any evidence of an essential singularity, as predicted for the PCH model (Kosterlitz and Thouless 1973, Kosterlitz 1974) and observed by the appropriate series analysis (Camp and Van Dyke 1975, Guttmann 1978).

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In three dimensions we find the susceptibility and specific heat both display conventional algebraic singularities, and we estimate the susceptibility exponent to be $\gamma = 1{\cdot}335 \pm 0{\cdot}010$ for both lattices, and the specific heat exponent to be $\alpha = -0{\cdot}04 \pm 0{\cdot}10$ for both lattices. This is in excellent agreement with the recent results of Rogiers *et al* (1978) on the three-dimensional $S = \frac{1}{2}$ $X-Y$ model.

The layout of this paper is as follows: in § 2 we define the model and outline the derivation of the series expansions. In § 3 the three-dimensional series are analysed, and in § 4 the two-dimensional series are discussed. Section 5 is devoted to a discussion and conclusion.

2. Derivation of series

The Hamiltonian of the model is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} C(\theta_i - \theta_j) - mH_z \sum_{i=1}^{N} C(\theta_i) \tag{2.1}$$

where

$$C(\phi) = \begin{cases} 1 & \text{for } |\phi| \leq \pi/2 \\ -1 & \text{for } \pi/2 < |\phi| \leq \pi \end{cases} \qquad \text{with } C(\phi + 2\pi) = C(\phi). \tag{2.2}$$

H_z is the z component of the magnetic field, m is the magnetic moment per spin, and the sum is over nearest neighbour pairs.

As shown in I, the zero-field partition function,

$$\begin{aligned} Z_N &= \int \cdots \int_{-\pi}^{\pi} \prod_{i=1}^{N} \left(\frac{\mathrm{d}\phi_i}{2\pi}\right) \prod_{\langle ij \rangle} \cosh K[1 + \tanh KC(\phi_i - \phi_j)] \\ &= (\cosh K)^{Nq/2} \left(1 + \sum_{\{G\}} W(G)L(G)v^n\right) \end{aligned} \tag{2.3}$$

where q is the coordination number of the lattice, $K = J/kT$, $v = \tanh K$ and the sum is over all single bond graphs with vertices of even degree. n is the number of bonds in the graph. $L(G)$ is a polynomial in N, and in the thermodynamic limit only those terms linear in N contribute to the free energy. $W(G)$ is the weight factor for graph G, and is defined by

$$W(G) = \int \cdots \int_{-\pi}^{\pi} \prod_{i=1}^{N} \left(\frac{\mathrm{d}\phi_i}{2\pi}\right) \prod_{\substack{\text{edges}\\ \text{of } G}} C(\phi_i - \phi_j). \tag{2.4}$$

The zero-field isothermal susceptibility is defined through the correlation function $\langle C(\theta_i - \theta_j) \rangle$ (see I for fuller discussion of this point) and we obtain

$$\frac{kT\chi_0(T)}{m^2} = 1 + 2 \sum_{\{G\}} v^n W(\tilde{G})L(G) \tag{2.5}$$

where the sum is over all magnetic graphs, that is, graphs with exactly two odd degree vertices. $L(G)$ is the weak lattice constant for graph G, and $W(\tilde{G})$ is the weight of the associated graph $\tilde{G}$ constructed from G by joining together by a single bond the two odd degree vertices in G. Any resultant double bonds are deleted.

We have tabulated all contributing zero-field partition function graphs up to and including order v^{20} (SQ), v^{16} (SC) and v^{14} (BCC). Similarly, we have tabulated all contributing magnetic graphs (contributing to the susceptibility) to order v^{12} (SQ and SC) and v^{11} (BCC). The magnetic graphs appear in Guttmann and Nymeyer (1977).

As discussed in I, the integral (2.4) defining the weight function $W(G)$ can be evaluated exactly for all contributing graphs with the exception of the type 17 graphs. For these graphs we have carried out the integration numerically, which results in a small uncertainty in some of the higher-order coefficients. These uncertainties are too small to affect the subsequent analysis. A full discussion of the evaluation of (2.4) for all contributing graphs is given in I.

Given the lattice constants and the weights, it is a straightforward matter to construct the specific heat and susceptibility series using (2.3) and (2.5) and carrying out the appropriate differentiations.

The resulting specific heat and susceptibility series are shown in tables 1 and 2 respectively.

Table 1. Specific heat series coefficients, $C_0/Nk = \Sigma_{n\geq 0}\, a_n v^n$.

N	SQ a_n	SC a_n	BCC a_n
0	0·000000000	0·00000000	0·00000000
2	2·00000000	3·00000000	4·00000000
4	3·33333333	11·00000000	14·6666667
6	−0·311111111	63·5333333	55·9377778
8	−6·85079365	356·257143	577·029841
10	−14·9210582	2245·91048	72797·9589
12	−17·4616899	15098·5055	96270·2 ± 0·4
14	−7·39761717	104794·270	13071510·0 ± 60·0
16	16·5290007	743090·0 ± 2·0	
18	48·9480566		
20	74·7227090		

Table 2. Zero-field isothermal susceptibility coefficients, $kT\chi_0(T)/Nm^2 = \Sigma_{n\geq 0}\, b_n v^n$.

N	SQ b_n	SC b_n	BCC b_n
0	1·00000000	1·00000000	1·00000000
1	4·00000000	6·00000000	8·00000000
2	6·00000000	15·0000000	28·0000000
3	12·00000000	50·0000000	130·666667
4	20·8333333	151·250000	551·666667
5	35·2000000	463·200000	2362·66667
6	59·4166667	1398·00833	9957·41111
7	94·4634921	4185·94286	41662·1460
8	152·082440	12451·9757	173665·711
9	234·350617	36818·0529	719314·563
10	362·146147	108779·910	2973821·20 ± 0·03
11	547·188815	319708·005	12243043·34 ± 0·35
12	819·803386	938717·92 ± 0·02	

3. Analysis of three-dimensional series

The analysis of the three-dimensional series was carried out using standard Padé and ratio method techniques (Gaunt and Guttmann 1974). Initially, Padé approximants to the logarithmic derivative of the susceptibility series were formed. The poles and residues of the diagonal and off-diagonal approximants are shown in tables 3 and 4 for the SC and BCC lattices respectively. From these tables we made the estimates

$$\begin{aligned} &\gamma = 1{\cdot}36 \pm 0{\cdot}06 \qquad v_c = 0{\cdot}351 \pm 0{\cdot}002 \qquad (\text{SC}) \\ &\gamma = 1{\cdot}36 \pm 0{\cdot}05 \qquad v_c = 0{\cdot}250 \pm 0{\cdot}001 \qquad (\text{BCC}). \end{aligned} \tag{3.1}$$

Table 3. Poles and residues of Padé approximants to the logarithmic derivative of the SC lattice susceptibility series.

N	$[N/(N-1)]$	$[N/N]$	$[N/(N+1)]$
3		0·3405 (−1·183)	0·3546 (−1·548)
4	0·3516 (−1·391)	0·3507 (−1·365)	0·3522 (−1·411)
5	0·3513 (−1·383)	0·3517 (−1·270)	0·3511 (−1·374)
6	0·3515 (−1·389)		

Table 4. Poles and residues of Padé approximants to the logarithmic derivative of the BCC lattice susceptibility series.

N	$[N/(N-1)]$	$[N/N]$	$[N/(N+1)]$
1			
2		0·2436 (−1·183)	0·2433 (−1·178)
3	0·2433 (−1·178)	0·2436* (−1·183*)	0·2522 (−1·435)
4	0·2502 (−1·351)	0·2504 (−1·360)	0·2509 (−1·379)
5	0·2500 (−1·347)	0·2506 (−1·370)	

Initial attempts at a direct ratio method analysis gave similar results, but with no significant decrease in the confidence limits on the estimate of γ and v_c. This lack of increased precision is due largely to the alternating trend of the ratios. This alternating trend is a characteristic of loose-packed lattices for this and other lattice models, and is due to a singularity at $v = -v_c$ if the critical point is at $v = v_c$. For this reason we applied a transformation to the series which maps the singularity at $-v_c$ to infinity. The transformation chosen was $x = 2v/(1 + v/v_c)$. This has the property that, if v_c is precisely known, then $x_c = v_c$, and the point $v = -v_c$ is mapped to infinity. Even if v_c is only known approximately, as in this case, the point $v = -v_c$ will be mapped far from the origin in the x plane, thereby substantially eliminating the effect of the singularity at $v = -v_c$.

Choosing $1/v_c = 4{\cdot}0$ (BCC) and $2{\cdot}86$ (SC), transformed series in the x variable were obtained. Such a transformation of course leaves the critical exponent, but not the critical amplitude, unchanged. The transformed series are shown in tables 5 and 6 for the SC and BCC susceptibility respectively, along with the ratios of successive coefficients and the linear and quadratic extrapolants of successive ratios. In the new variable, all these sequences are monotone, and can be extrapolated with confidence.

Table 5. Ratio analysis of transformed SC lattice susceptibility series.

N	Coefficients a_n	Ratios $r_n = a_n/a_{n-1}$	Linear extrapolants $s_n = nr_n - (n-1)r_{n-1}$	Quadratic extrapolants $t_n = [ns_n - (n-2)s_{n-1}]/2{\cdot}0$
0	1·00000000			
1	3·00000000	3·00000		
2	8·0400000	2·68000	2·3600	
3	23·10970	2·87434	3·2630	3·7145
4	68·04337	2·94436	3·1544	3·3229
5	201·6386	2·96338	3·0395	2·7477
6	597·7545	2·96448	2·9700	2·7951
7	1769·125	2·95962	2·9304	2·8321
8	5224·239	2·95301	2·9067	2·8427
9	15391·656	2·94620	2·8917	2·8464
10	45247·876	2·93977	2·8819	2·8496
11	132752·55	2·93390	2·8752	2·8526
12	388781·17	2·92862	2·8706	2·8548

Table 6. Ratio analysis of transformed BCC lattice susceptibility series.

N	Coefficients a_n	Ratios $r_n = a_n/a_{n-1}$	Linear extrapolants $s_n = nr_n - (n-1)r_{n-1}$	Quadratic extrapolants $t_n = [ns_n - (n-2)s_{n-1}]/2{\cdot}0$
0	1·0			
1	4·0	4·0		
2	15·0	3·75	3·5000	
3	60·33333	4·02222	4·5667	5·1000
4	248·4792	4·11844	4·4071	4·2475
5	1029·667	4·14388	4·2456	4·0034
6	4267·751	4·14479	4·1494	3·9568
7	17659·17	4·13782	4·0960	3·9625
8	72909·95	4·12873	4·0651	3·9726
9	300356·6	4·11956	4·0462	3·9797
10	1234760·0	4·11098	4·0338	3·9844
11	5066472·0	4·10320	4·0254	3·9877

Simply taking the mean of the last entry in the sequence of linear and quadratic extrapolants, which are separately monotone decreasing and monotone increasing respectively, gives $1/x_c = 2{\cdot}8591 \pm 0{\cdot}0115$ (SC) and $1/x_c = 4{\cdot}0066 \pm 0{\cdot}0189$ (BCC). However a more careful study of these sequences shows that the linear extrapolants are decreasing more rapidly than the quadratic extrapolants are increasing. Taking this trend into account produces the refined estimates $1/x_c = 2{\cdot}8575 \pm 0{\cdot}0025$ (SC) and $1/x_c = 3{\cdot}9995 \pm 0{\cdot}0035$ (BCC). Transforming back to the original variable, we obtain $1/v_c = 2{\cdot}8550 \pm 0{\cdot}0025$ (SC) and $1/v_c = 3{\cdot}9990 \pm 0{\cdot}0035$ (BCC).

These estimates of the critical point are used for the subsequent analysis. First, Padé approximants were formed to $(v_c - v)(\mathrm{d}/\mathrm{d}v)\log\chi(v)|_{v=v_c}$, in order to estimate the critical exponent γ. The resulting approximants are shown in table 7. For both

Table 7. Padé approximants to $(v_c - v)\,\mathrm{d}(\log\chi_0(v))/\mathrm{d}v|_{v=v_c}$ for the SC and BCC lattices. v_c is given in the text.

N	$[N/(N-1)]$	$[N/N]$	$[N/(N+1)]$
SC			
2			1·379
3		1·367	1·403
4	1·354	1·353	1·339
5	1·355	1·334	1·349
6	1·325		
BCC			
2			1·361
3		1·355	1·374
4	1·347	1·344	1·336
5	1·348	1·336	

lattices there is a slight decreasing trend with increasing order of approximant. The results however are very similar for both lattices, and we estimate $\gamma = 1{\cdot}33 \pm 0{\cdot}02$ for both. If one seeks a simple fraction, $\gamma = 4/3$ seems the most likely. In any event, it is a useful mnemonic. Since the transformed susceptibility series essentially eliminates the effect of the singularity at $-v_c$, it was considered worthwhile to study Padé approximants to $(x_c - x)(\mathrm{d}/\mathrm{d}x)\log\chi^*(x)$, where $\chi^*(x)$ is the transformed susceptibility series. There is a widespread misconception that, since diagonal Padé approximants are invariant under homographic transformations, such calculations are useless. However the operation of homographically transforming a function and taking its logarithmic derivative are not commutative, so the eponymous transformation is far from useless in this situation. In table 8 we show Padé approximants to $(x_c - x)(\mathrm{d}/\mathrm{d}x)\log\chi^*(x)|_{x=x_c}$ which should yield estimates of critical exponents. These approximants are rather better converged than those derived from the series in the original variable, and we make the estimate $\gamma = 1{\cdot}335 \pm 0{\cdot}010$ for both lattices. For subsequent investigations we can confidently use the estimate $\gamma = 4/3$. Using this

Table 8. Padé approximants to $(x_c - x)\,\mathrm{d}(\log\chi_0^*(x))/\mathrm{d}x|_{x=x_c}$, where $\chi_0^*(x)$ is the transformed susceptibility series, for the SC and BCC lattices. x_c is given in the text.

N	$[N/(N-1)]$	$[N/N]$	$[N/(N+1)]$
SC			
2			1·379
3		1·372	1·403
4	1·361	1·331	1·339
5	1·338	1·337	1·338
6	1·338	1·349	
BCC			
2			1·361
3		1·358	1·374
4	1·352	1·330	1·336
5	1·335	1·336	

value, we perform the usual consistency test, by forming Padé approximants to $(\chi(v))^{1/\gamma}$, which should have poles at $1/v_c$. These approximants are shown in table 9, and are indeed consistent with our previous estimates of v_c. However the ratio method analysis of the transformed series appears to give better converged estimates of v_c, so that this consistency test does not improve the accuracy of our estimate of v_c, as if often does.

Table 9. Padé approximants to $(\chi_0(v))^{1/\gamma}$, with $\gamma = 4/3$, for the SC and BCC lattices.

N	$[N/(N-1)]$	$[N/N]$	$[N/(N+1)]$
SC			
3		0·3491	0·3485
4	0·3486	0·3489	0·3496
5	0·3504	0·3512	0·3502
6	0·3506	0·3504	
BCC			
2			0·2422
3		0·2497	0·2493
4	0·2494	0·2495	0·2498
5	0·2500	0·2513	0·2500
6	0·2502		

We have also performed a ratio method consistency check, by calculating biased exponent estimates. That is, using the estimates of the critical point, the sequence $\{r_n/(1+g/n)\}$ was investigated, where $r_n = a_n/a_{n-1}$ is the ratio of successive susceptibility coefficients, and $g = \gamma - 1$ was taken to be 1/3. The resultant sequence was appropriately extrapolated for a loose-packed lattice, that is, alternate pairs were linearly and quadratically extrapolated against $1/n$. To conserve space we do not give the detailed numerical data, but merely remark that the results are entirely consistent with the earlier, unbiased, estimates of v_c.

In order to estimate the critical amplitudes, we used the cited estimates of γ and v_c, and formed Padé approximants to $(v_c - v)\chi(v)^{1/\gamma}|_{v=v_c}$. If we write $\chi(v) \sim A(1-v/v_c)^{-\gamma}$, then these Padé approximants should converge to $A^{1/\gamma}v_c$. The approximants are shown in table 10, from which we estimate $v_cA^{1/\gamma} = 0{\cdot}408 \pm 0{\cdot}001$ (SC), $v_cA^{1/\gamma} = 0{\cdot}2775 \pm 0{\cdot}0015$ (BCC) so that $A = 1{\cdot}121 \pm 0{\cdot}003$ (SC) and $A = 1{\cdot}081 \pm 0{\cdot}004$ (BCC). It is customary to express amplitudes directly in terms of the temperature variable $t = T_c/T - 1$. Writing $\chi(T) \sim C^+ t^{-\gamma}$, it follows from the above estimates of A that $C^+ = 1{\cdot}124 \pm 0{\cdot}004$ (SC) and $C^+ = 1{\cdot}059 \pm 0{\cdot}005$ (BCC).

Turning now to the specific heat series for the SC and BCC lattices, we show in table 11 the ratios ($r_n = a_{2n}/a_{2n-2}$) and linear extrapolants ($t_n = nr_n - (n-1)r_{n-1}$) of the specific heat coefficients, defined by $C/Nk = \Sigma_{n\geq 0}\, a_n v^n$. These series are expansions in v^2, since the lattice topology only admits the embedding of non-magnetic graphs with an even number of bonds. Thus the series are expected to diverge at v_c^2, as the specific heat and susceptibility must have the same critical temperature since they both derive from a single partition function, unless there is more than one phase transition point. The results shown in table 11 are consistent with the expected values of the limits of the sequences $\{t_n\}$, and for subsequent analyses we will assume these values of the critical point.

Table 10. Padé approximants to $(v_c - v)[\chi_0(v)]^{1/\gamma}|_{v=v_c}$ for the SC and BCC lattices. $\gamma = 4/3$ and v_c given in the text.

N	$[N/(N-1)]$	$[N/N]$	$[N/(N+1)]$
SC			
3		0·4031	0·4037
4	0·3992	0·4067	0·4090
5	0·4082	0·4078	0·4079
6	0·4079	0·4078	
BCC			
2		0·2716	0·2761
3	0·2748	0·2753	0·2757
4	0·2745	0·2768	0·2780
5	0·2776	0·2773	0·2774
6	0·2774		

Table 11. Ratio method analysis of specific heat series for the SC and BCC lattices to determine v_c and α.

	SC			BCC		
N	r_n	t_n	β_n	r_n	t_n	β_n
2	3·66667		−1·1003	3·66667		
3	5·77576	9·99395	−0·8742	38·1393	346·377	
4	5·60741	5·10236	−1·2482	10·3156	−73·1555	−1·4198
5	6·30418	9·09136	−1·1329	12·6160	21·8176	−1·0555
6	6·72267	8·81490	−1·0514	13·2243	16·2658	−1·0384
7	6·94070	8·24900	−1·0394	13·5780	15·6999	−1·0567
8	7·09094	8·14262	−1·0404	⋮		
⋮	⋮	⋮				
∞						
	Expected limit $= 1/v_c^2 = 8{\cdot}151$			Expected limit $= 1/v_c^2 = 15{\cdot}992$		

Writing the specific heat as $C_0(v^2)/Nk \sim F(1 - v^2/v_c^2)^{-\alpha}$, we attempted to estimate the critical exponent α by forming sequences $\{\beta_n\}$ where $\beta_n = n(r_n v_c^2 - 1) \to \alpha - 1$. These are also shown in table 11. While rather short, the sequences $\{\beta_n\}$ do appear to have settled down to a value close to -1 for both lattices. If this value were precisely -1, this would correspond to a logarithmic singularity. In fact, we estimate $\alpha = -0{\cdot}04 \pm 0{\cdot}1$ for both lattices. Such weakly singular specific heat series are notoriously difficult to analyse. Padé approximants to $(v_c^2 - v^2)\,\mathrm{d}(C_0(v^2))/\mathrm{d}v^2|_{v=v_c}$ (not shown) only yield the estimates $\alpha < 0{\cdot}2$ (SC) and $\alpha < 0{\cdot}3$ (BCC). This is typical of the behaviour of Padé approximants for such series. Our earlier estimate (I) for the FCC lattice was $\alpha = -0{\cdot}04 \pm 0{\cdot}06$, in agreement with the value obtained here for the SC and BCC lattices. This value of α in fact corresponds to a cusp-like singularity, but numerically α is so small, and the series so short, that $\alpha = 0{\cdot}00$ cannot be ruled out, and is indeed the nearest simple fraction. In order to analyse the series for the specific heat amplitude, we have assumed a logarithmic divergence, that is, we have assumed that $C_0(T) \sim A^+ \ln(1-T_c/T)$. The value of the critical amplitude A^+ can be estimated by forming Padé approximants to $(T_c - T)\,dC_0(T)/dT|_{T=T_c}$. In this way we obtained

$$A^+ = -0{\cdot}29 \pm 0{\cdot}01 \text{ (sc)}; \qquad A+ = -0{\cdot}31 \pm 0{\cdot}02 \text{ (bcc)}.$$

4. Two-dimensional series

The Padé and ratio method analysis described in the previous section has also been applied to the two-dimensional square lattice series, and we find no sign of a conventional algebraic singularity on the positive T axis. Analysis for an essential singularity of the type suggested by Kosterlitz (1974) for the PCH model has also been attempted, using the method described in Guttmann (1978). No evidence of that type of singularity was obtained either.

Neither of these results are surprising. Since this model has the planar symmetry of the PCH model, it is expected to show non-simple behaviour in two dimensions. In particular since the PCH model does not admit long-range order for $T > 0$ and appears not to have a conventional algebraic singularity, these properties are also expected in the step model. For the PCH model however, vortex excitations (Kosterlitz 1974) lead to an essential singularity for both the susceptibility and specific heat. Such excitations are not however expected for the step model, since interacting spins with angular separation of less than $\pi/2$ are energetically equivalent to parallel spins, that is, zero-energy vortices can be formed.

An additional complication is that the susceptibility has been calculated via the correlation function $\langle C(\theta_i)C(\theta_j)\rangle$, whereas the normal Zeeman spin–field interaction leads to a susceptibility defined via the correlation function $\langle S_i \,.\, S_j\rangle$. This point is discussed in some detail in I. As mentioned there, this choice of correlation function is unlikely to affect the behaviour of the susceptibility in the three-dimensional system. However in two dimensions, where the very existence of a phase transition is such a delicate matter, it is necessary to be rather more guarded. Thus the susceptibility behaviour we observe *may* be an artifact of our choice of correlation function. No such reservation applies in the zero-field specific heat calculation however, where the spin–field interaction plays no part. The nature of the phase transition—if any—for this model in two dimensions thus remains an intriguing open question.

5. Discussion and conclusion

In three dimensions this model displays critical behaviour in precise agreement with that observed by Rogiers *et al* (1978) for the spin-$\frac{1}{2}$ X–Y model. It is in trifling disagreement with the earlier result of Bowers and Joyce (1967) for the PCH model, who obtained $\gamma = 1{\cdot}312 \pm 0{\cdot}006$. It is not unlikely that longer series would bring their estimate into agreement with that obtained for this model, $\gamma = 1{\cdot}335 \pm 0{\cdot}010$ and for the $S = \frac{1}{2}$ X–Y model, $\gamma = 1{\cdot}333 \pm 0{\cdot}001$. The specific heat exponent $\alpha = -0{\cdot}04 \pm 0{\cdot}10$ though not precisely estimated, is in agreement with the spin-$\frac{1}{2}$ X–Y model estimates $\alpha = 0{\cdot}00 \pm 0{\cdot}04$ of Rogiers *et al* (1978), and the PCH model estimates of Bowers and Joyce (1967), who obtained $0 \leqslant \alpha \leqslant \frac{1}{32}$. We also find agreement with the earlier estimates (I) on the FCC lattice $\gamma = 1{\cdot}32 \pm 0{\cdot}04$ and $\alpha = -0{\cdot}04 \pm 0{\cdot}06$.

Thus in three dimensions the evidence is strong that the critical exponents are determined by spin-space and lattice dimensionality for a variety of systems with finite-range interaction.

For the two-dimensional model, as discussed in the previous section, we have been unable to elucidate the nature of the phase transition. (This point is also discussed in I.) However, there is no evidence of either a conventional algebraic singularity or a PCH-like essential singularity. It may be that there is no phase transition in two dimensions, or that the phase transition is of more complex type than that observed for either the PCH or the Ising model. For the two-dimensional $S=\frac{1}{2}$ $X-Y$ model, Betts *et al* (1971) found evidence of an algebraically divergent susceptibility with an exponent of 1·5. Recent substantial extensions to this series have been obtained, but the nature of the singularity is now less clear cut (D D Betts, private communication).

For the one-dimensional system, we take this opportunity to correct a minor error in I, and give the susceptibility in the thermodynamic limit as

$$\frac{kT\chi_0(T)}{m^2}=1+2\sum_{j=2}^{\infty}\langle C(\theta_i)C(\theta_j)\rangle=1+\frac{4\tan v/2}{1-\tan v/2}. \tag{5.1}$$

Finally, we note that we have insufficient data to test lattice–lattice scaling (Betts *et al* 1971) for this model, though we do have enough data to use it if we assume its applicability. Assuming the usual two-parameter scaling laws, all the other critical exponents follow immediately and assuming lattice–lattice scaling yields all the critical amplitudes for the other thermodynamic quantities.

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